ELSEVIER

Journal of Geometry and Physics 23 (1997) 259–286

JOURNAL OF GEOMETRY AND PHYSICS

On nearly parallel G_2-structures

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Abstract

A nearly parallel G_2-structure on a seven-dimensional Riemannian manifold is equivalent to a spin structure with a Killing spinor. We prove general results about the automorphism group of such structures and we construct new examples. We classify all nearly parallel G_2-manifolds with large symmetry group and in particular all homogeneous nearly parallel G_2-structures.

Subj. Class.: Differential geometry

1991 MSC: 53C25, 57R15

Keywords: G_2-structure; Sasakian manifold; Killing spinor

Contents

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SSDI 0393-0440(95)00050-X

1. Introduction

A nearly parallel G_2-structure on a seven-dimensional manifold is a 3-form ω^3 of special algebraic type satisfying the differential equation

$$d\omega^3 = -8\lambda(*\omega^3)$$

for some constant $\lambda \neq 0$. The existence of ω^3 is equivalent to the existence of a spin structure with a Killing spinor, i.e. a spinor ψ satisfying

$$\nabla_X \psi = \lambda X \cdot \psi \quad \forall X \in TM.$$

In case $\lambda = 0$, ω^3 defines a *geometric* G_2-structure ($d\omega^3 = 0$, $\delta\omega^3 = 0$). Excluding the case of the seven-dimensional sphere there are three types of nearly parallel G_2-structures depending on the dimension of the space KS of all Killing spinors. Nearly parallel G_2-structures with $\dim(KS) = 3$ are 3-Sasakian manifolds and nearly parallel G_2-structures such that $\dim(KS) = 2$ are Einstein–Sasakian spaces. There are examples of compact nearly parallel G_2-manifolds where the dimension of the space of all Killing spinors equals one and we call such spaces proper G_2-manifolds.

Recently Joyce [22] solved an open problem in holonomy theory, namely the existence problem of compact seven-dimensional Riemannian manifolds with G_2-holonomy. On the other hand, Boyer et al. [7] constructed new compact examples of 3-Sasakia manifolds and investigated the global geometry of these spaces. In dimension seven, 3-Sasakian manifolds are special nearly parallel G_2-structures and such manifolds have been studied a long time ago (see [12,20]). However, during the last 10 years, these special Einstein manifolds appeared as Einstein spaces where the Dirac operator has the smallest possible eigenvalue and many compact examples are known since this time (see [11]). The aim of this paper is to revisit once again the results as well as the examples of compact nearly parallel G_2-structures known up to now. Moreover, starting from 3-Sasakian manifold we construct new manifolds with a nearly parallel G_2-structure. A 3-Sasakian manifold admits a second Einstein metric obtained from the given one by scaling the metric in the directions of the orbits of the $Spin(3)$-action. It turns out that this Einstein metric is a proper G_2-structure and we obtain new nearly parallel G_2-structures from the examples of 3-Sasakian manifolds mentioned above.

Finally we investigate the automorphism group of a compact nearly G_2-manifold and we classify in particular all homogeneous G_2-manifolds. The automorphism group $G = Aut(M^7, \omega^3)$ of a nearly parallel G_2-manifold has some special properties. In particular, if $\dim(G) \geq 10$, G acts transitively on M^7. The zero set of infinitesimal automorphisms is either one- or three-dimensional and a four-dimensional orbit of this group action is of special topological and geometric type. Moreover, the isotropy groups $G(m)$ are subgroups of the exceptional G_2 and one can list them explicitly. Combining all these informations we can classify the compact, nearly parallel G_2-manifolds with a large symmetry group.

2. The exceptional group G_2

The group G_2 is a compact, simple and simply connected 14-dimensional Lie group. In this section we collect some basic algebraic facts about this group. In particular, we will define G_2 as the isotropy group of a real $Spin(7)$-spinor. Since in dimension seven these spinors correspond to the 3-forms ω^3 of general type in $\Lambda^3(\mathbb{R}^7)$, this definition of the group G_2 is equivalent to the usual one as the subgroup of $GL(7;\mathbb{R})$ preserving the 3-form in $\mathbb{R}^7$

$$\begin{aligned}\omega_0^3 = {} & e_1\wedge e_2\wedge e_7 + e_1\wedge e_3\wedge e_5 - e_1\wedge e_4\wedge e_6\\ & - e_2\wedge e_3\wedge e_6 - e_2\wedge e_4\wedge e_5 + e_3\wedge e_4\wedge e_7 + e_5\wedge e_6\wedge e_7. \end{aligned}\tag{1}$$

The advantage of this point of view is that a topological G_2-structure on a seven-dimensional manifold defines a Riemannian metric as well as a spinor field of constant length. We shall use the equivalence between topological G_2-structures and 3-forms of general type and between these and Riemannian metrics together with a unit spinor field many times in our investigations of G_2-structures of special geometrical type.

Let $e_1,\ldots,e_7$ be the standard orthonormal basis of the Euclidian vector space $\mathbb{R}^7$ and denote by $Cliff(\mathbb{R}^7)$ the real Clifford algebra. We will use the real representation of this algebra on $\Delta_7 := \mathbb{R}^8$ given on its generators by

$$\begin{aligned}
e_1 &= E_{18} + E_{27} - E_{36} - E_{45}, & e_2 &= -E_{17} + E_{28} + E_{35} - E_{46},\\
e_3 &= -E_{16} + E_{25} - E_{38} + E_{47}, & e_4 &= -E_{15} - E_{26} - E_{37} - E_{48},\\
e_5 &= -E_{13} - E_{24} + E_{57} + E_{68}, & e_6 &= E_{14} - E_{23} - E_{58} + E_{67},\\
e_7 &= E_{12} - E_{34} - E_{56} + E_{78},
\end{aligned}$$

where E_{ij} is the standard basis of the Lie algebra $\mathfrak{so}(8)$:

$$E_{ij} = \begin{array}{cl} \begin{pmatrix} 0 & \cdots\cdots\cdots & 0\\ \cdots\cdots\cdots & -1 & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots & 1 & \cdots\cdots\cdots \\ 0 & \cdots\cdots\cdots & 0 \end{pmatrix} & \begin{array}{l} \\ \cdots\ i \\ \\ \cdots\ j \\ \\ \end{array} \\ \quad\ \vdots \qquad\quad \vdots & \\ \quad\ i \qquad\quad j & \end{array}$$

If we restrict this representation to $Spin(7) \subset Cliff(\mathbb{R}^7)$ we obtain the real spin representation $\kappa : Spin(7) \longrightarrow SO(\Delta_7)$. The group $Spin(7)$ acts transitively on the sphere

$$S(\Delta_7) = \{\|\psi\| = 1\} \subset \Delta_7 = \mathbb{R}^8.$$

We now define the group G_2 as the subgroup of $Spin(7)$ preserving the spinor $\psi_0 := {}^t(1,0,\ldots,0)$

$$G_2 = \{g \in Spin(7) \mid g\psi_0 = \psi_0\}.$$

Consequently the sphere S^7 is diffeomorphic to the homogeneous space $Spin(7)/G_2$ and we obtain from the exact homotopy sequence of this fibration

$$\pi_0(G_2) = 0, \quad \pi_1(G_2) = 0, \quad \pi_2(G_2) = 0, \quad \pi_3(G_2) = \mathbb{Z}.$$

Let us now calculate the Lie algebra $\mathfrak{g}_2$ of G_2. We identify the Lie algebra of $Spin(7)$ with $\mathfrak{spin}(7) = \{\omega = \sum_{i<j} \omega_{ij} e_i e_j \mid \omega_{ij} \in \mathbb{R}\} \subset \mathit{Cliff}(\mathbb{R}^7)$. The Lie algebra $\mathfrak{g}_2$ is the subalgebra of this algebra containing all elements ω satisfying $\omega \cdot \psi_0 = 0$. Let $\omega = \sum_{i<j} \omega_{ij} e_i e_j$ be any element of $\mathfrak{spin}(7)$. Then $\omega \cdot \psi_0 = 0$ holds iff

$$\begin{array}{ccc} \omega_{12} + \omega_{34} + \omega_{56} = 0, & -\omega_{13} + \omega_{24} - \omega_{67} = 0, & -\omega_{14} - \omega_{23} - \omega_{57} = 0, \\ -\omega_{16} - \omega_{25} + \omega_{37} = 0, & \omega_{15} - \omega_{26} - \omega_{47} = 0, & \omega_{17} + \omega_{36} + \omega_{45} = 0, \\ \omega_{27} + \omega_{35} - \omega_{46} = 0. & & \end{array}$$

We consider the universal covering $Spin(7) \longrightarrow SO(7)$ of the special orthogonal group $SO(7)$. Because of $(-1) \notin G_2$, there is an isomorphism from G_2 onto a subgroup of $SO(7)$, which we also denote by G_2. We now describe this group. This will yield a second definition of G_2 using 3-forms on $\mathbb{R}^7$. The key point is a special relation in dimension seven between real spinors and generic 3-forms.

Let $\psi \in \Delta_7$ be a fixed spinor. Then the map

$$\mathbb{R}^7 \ni X \longmapsto X\psi \in \Delta_7$$

is an isomorphism between $\mathbb{R}^7$ and the orthogonal complement of ψ in Δ_7. We observe that for $X, Y \in \mathbb{R}^7$ the spinors ψ and $YX\psi + \langle X, Y\rangle\psi$ are orthogonal to each other. Therefore we can define a (2, 1)-tensor A_ψ by

$$YX\psi = -\langle X, Y\rangle\psi + A_\psi(Y, X)\psi. \tag{2}$$

A_ψ has the following properties:
(1) $A_\psi(X, Y) = -A_\psi(Y, X)$,
(2) $\langle Y, A_\psi(Y, X)\rangle = 0$,
(3) $A_\psi(Y, A_\psi(Y, X)) = -\|Y\|^2 X + \langle X, Y\rangle Y$.
It defines a 3-form ω^3_ψ by $\omega^3_\psi(X, Y, Z) = \langle X, A_\psi(Y, Z)\rangle$.

Vice versa, a (2, 1)-tensor A on $\mathbb{R}^7$ which has properties (1)–(3) defines a one-dimensional subspace $E(A) = \{\psi \in \Delta_7 \mid YX\psi = -\langle X, Y\rangle\psi + A(Y, X)\psi\}$. Consequently, we obtain a bijection from the projective space $P(\Delta_7) = \mathbb{R}\mathrm{P}^7$ onto the set of 3-forms $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ whose tensor A defined by $\omega^3(X, Y, Z) = \langle X, A(Y, Z)\rangle$ has the above-mentioned properties.

In particular, if $\psi = \psi_0 := {}^t(1, 0, \ldots, 0)$, then a direct calculation yields $\omega^3_{\psi_0} = \omega^3_0$, where ω^3_0 is given by Eq. (1).

Let g be an element of $Spin(7)$ and $\pi(g)$ the corresponding element in $SO(7)$. We compare the 3-forms associated to the spinors ψ and $g\psi$ and obtain the equation

$$\omega^3_{g\psi} = (\pi(g^{-1}))^* \omega^3_\psi.$$

The 3-form ω^3_ψ defines the spinor ψ up to a real number. Hence, the image of the group $G_2 \subset Spin(7)$ with respect to $\pi : Spin(7) \longmapsto SO(7)$ equals

$$G_2 = \{A \in SO(7) \mid A^*\omega^3_{\psi_0} = \omega^3_{\psi_0}\}.$$

However, the equation $A^*\omega^3_0 = \omega^3_0$ for $A \in GL(7)$ implies $A \in SO(7)$. See for a proof [8,24]. Using this, we obtain

$$G_2 = \{A \in GL(7) \mid A^*\omega^3_0 = \omega^3_0\}.$$

Remark 2.1. Similarly, we can investigate the action of *Spin*(7) on the Stiefel manifolds $V_2(\Delta_7)$ and $V_3(\Delta_7)$ of orthonormal pairs and triples of spinors, respectively. This action is transitive, too. The isotropy group of a fixed pair of spinors is isomorphic to $SU(3)$ and the one of a triple is isomorphic to $SU(2)$.

Remark 2.2. The $GL(7)$-orbit $\Lambda^3_+(\mathbb{R}^7) := \{A^*\omega^3_0 \mid A \in GL(7)\}$ is an open subset of $\Lambda^3(\mathbb{R}^7)$ since $\dim \Lambda^3(\mathbb{R}^7) = 35$ and $\dim GL(7) - \dim G_2 = 49 - 14 = 35$. Let α^3 be an element of this orbit, i.e. $\alpha^3 = A^*\omega^3$ for some $A \in GL(7)$. Then α^3 defines an inner product on $\mathbb{R}^7$ by $\langle\ ,\ \rangle_\alpha := A^*\langle\ ,\ \rangle$, an orientation $O_\alpha := A^*(e_1 \wedge \cdots \wedge e_7)$ and a corresponding Hodge operator $*_\alpha : \Lambda^p(\mathbb{R}^7) \longmapsto \Lambda^{7-p}(\mathbb{R}^7)$.

Remark 2.3. Let $\psi_1, \psi_2 \in \Delta_7$ be spinors of the same length and $\xi \in \mathbb{R}^7$ such that $\xi\psi_1 = \psi_2$. Then we have for the induced 3-forms $\omega^3_1 = \omega^3_{\psi_1}$ and $\omega^3_2 = \omega^3_{\psi_2}$

$$\omega_2 = -\omega_1 + 2(\xi \lrcorner \omega_1) \wedge \xi.$$

In order to prove Remark 2.3, we use the equations which define the tensors $A_1 = A_{\psi_1}$ and $A_2 = A_{\psi_2}$. From $YX\psi_2 = -\langle Y, X\rangle\psi_2 + A_2(Y, X)\psi_2$ it follows that $YX\xi\psi_1 = -\langle Y, X\rangle\xi\psi_1 + A_1(Y, X)\xi\psi_1$. By the definition of A_1 this is equivalent to

$$\begin{aligned}
&-\langle X, \xi\rangle Y\psi_1 - \langle Y, A_1(X, \xi)\rangle\psi_1 + A_1(Y, A_1(X, \xi))\psi_1 \\
&\quad = -\langle Y, X\rangle\xi\psi_1 - \langle A_2(Y, X), \xi\rangle\psi_1 + A_1(A_2(Y, X), \xi)\psi_1,
\end{aligned}$$

or to

$$\begin{aligned}
&(-\langle X, \xi\rangle Y + A_1(Y, A_1(X, \xi)) + \langle Y, X\rangle\xi - A_1(A_2(Y, X), \xi))\psi_1 \\
&\quad = (\langle Y, A_1(X, \xi)\rangle + \langle A_1(Y, X), \xi\rangle)\psi_1.
\end{aligned}$$

Since the Clifford multiplication of real spinors by a vector is anti-symmetric we conclude that

$$A_1(Y, A_1(X, \xi)) + \langle Y, X\rangle\xi = A_1(A_2(Y, X), \xi) + \langle X, \xi\rangle Y, \tag{3}$$

$$\langle Y, A_1(X, \xi)\rangle = \langle A_2(Y, X), \xi\rangle, \tag{4}$$

where (4) is equivalent to $\omega_1(X, Y, \xi) = \omega_2(X, Y, \xi)$ and to $A_1(X, \xi) = A_2(X, \xi)$.

Let now $X, Y, Z \in \mathbb{R}^7$ be vectors orthogonal to ξ. There exists an $X \in \mathbb{R}^7$, $X \perp \xi$ such that $Z = A_1(X, \xi) = A_2(X, \xi)$. From Eqs. (3) and (4) we conclude

$$\begin{aligned}\langle W, A_1(Y, Z)\rangle &= \langle W, A_1(Y, A_1(X, \xi))\rangle = \langle W, A_1(A_2(Y, X), \xi)\rangle \\ &= \langle W, A_2(A_2(Y, X), \xi)\rangle = -\langle W, A_2(\xi, A_2(Y, X))\rangle,\end{aligned}$$

where the last equation holds because of property (3) of the (2, 1)-tensor A_2. Consequently, we get $\omega_1(W, Y, Z) = -\omega_2(W, Y, Z)$. The assertion follows.

Now we recall the decomposition of $\Lambda^p(\mathbb{R}^7)$ into irreducible components with respect to the action of G_2.

Proposition 2.4.
(1) *$\mathbb{R}^7 = \Lambda^1(\mathbb{R}^7) =: \Lambda^1_7$ is irreducible.*
(2) *$\Lambda^2(\mathbb{R}^7) = \Lambda^2_7 \oplus \Lambda^2_{14}$, where*

$$\begin{aligned}\Lambda^2_7 &= \{\alpha^2 \in \Lambda^2 \mid *(\omega^3 \wedge \alpha^2) = 2\alpha^2\} = \{X \lrcorner\, \omega^3 \mid X \in \mathbb{R}^7\}, \\ \Lambda^2_{14} &= \{\alpha^2 \in \Lambda^2 \mid *(\omega^3 \wedge \alpha^2) = -\alpha^2\} = \mathfrak{g}_2.\end{aligned}$$

(3) *$\Lambda^3(\mathbb{R}^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where*

$$\begin{aligned}\Lambda^3_1 &= \{t\omega^3 \mid t \in \mathbb{R}^1\}, \qquad \Lambda^3_7 = \{*(\omega^3 \wedge \alpha^1) \mid \alpha^1 \in \Lambda^1_7\}, \\ \Lambda^3_{27} &= \{\alpha^3 \in \Lambda^3 \mid \alpha^3 \wedge \omega^3 = 0, \alpha^3 \wedge *\omega^3 = 0\}.\end{aligned}$$

Proposition 2.5. *The wedge product $\omega^3\wedge : \Lambda^3(\mathbb{R}^7) \longrightarrow \Lambda^6(\mathbb{R}^7)$ has the following properties with respect to the decomposition $\Lambda^3(\mathbb{R}^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$:*
(1) *$\omega^3 \wedge (\Lambda^3_1 \oplus \Lambda^3_{27}) = 0$;*
(2) *if $\eta^3 = *(\omega^3 \wedge \alpha^1) \in \Lambda^3_7$, then $\omega^3 \wedge \eta^3 = -4 * \alpha^1$.*

Similarly, the wedge product $*\omega^3\wedge : \Lambda^2(\mathbb{R}^7) \longrightarrow \Lambda^6(\mathbb{R}^7)$ has the following properties with respect to the decomposition $\Lambda^2(\mathbb{R}^7) = \Lambda^2_7 \oplus \Lambda^2_{14}$:
(3) *$(*\omega^3) \wedge \Lambda^2_{14} = 0$;*
(4) *if $\alpha^2 = X \lrcorner\, \omega^3 \in \Lambda^2_7$, then $(*\omega^3) \wedge \alpha^2 = 3(*X)$.*

Next we study the action of the group G_2 on the Grassmannian manifolds $G_2(\mathbb{R}^7)$ and $G_3(\mathbb{R}^7)$ of oriented two- and three-dimensional linear subspaces in $\mathbb{R}^7$.

Proposition 2.6.
(1) *G_2 acts transitively on $G_2(\mathbb{R}^7)$.*
(2) *G_2 acts on $G_3(\mathbb{R}^7)$ with cohomogeneity one. The principal orbits have dimension* 11 *and there are two exceptional orbits of dimension eight.*
(3) *For any $E^3 \in G^3(\mathbb{R}^7)$ the inequality $|\omega^3(E^3)| \leq 1$ holds . The three-dimensional subspace E^3 belongs to the exceptional orbit with respect to the G_2-action if and only if $|\omega^3(E^3)| = 1$.*

Proof. $G_2(\mathbb{R}^7) = SO(7)/[SO(2) \times SO(5)]$ is a 10-dimensional manifold. On the other hand, the intersection of the Lie algebras $\mathfrak{g}_2$ and $\mathfrak{so}(2) \times \mathfrak{so}(5)$ is the four-dimensional subalgebra of $\mathfrak{so}(7)$ defined by the equations

$$\omega_{1i} = \omega_{2i} = 0 \text{ for } i \geq 3, \quad \omega_{57} = \omega_{37} = \omega_{47} = 0,$$

$$\omega_{36} + \omega_{45} = \omega_{35} - \omega_{46} = \omega_{24} - \omega_{67} = 0, \quad \omega_{12} + \omega_{34} + \omega_{56} = 0.$$

Hence the G_2-orbit of the standard 2-plane $Span\{e_1, e_2\}$ has dimension 10. Since this orbit is a compact submanifold of $G_2(\mathbb{R}^7)$, it coincides with the Grassmannian manifold.

Fix a three-dimensional subspace E^3. Since G_2 acts transitively on $G_2(\mathbb{R}^7)$ in the G_2-orbit through E^3 there exists a three-dimensional subspace containing the vectors e_1 and e_2. For simplicity we also denote this space by E^3. The isotropy group of the vectors e_1, e_2 inside G_2 is the group $SU(2)$ acting on $Span\{e_3, e_4, e_5, e_6\}$. Therefore we may assume that the third vector of E^3 is given by $\cos(\varphi)e_3 + \sin(\varphi)e_7$. Consequently, any G_2-orbit in $G_3(\mathbb{R}^7)$ contains a subspace of the special form

$$E^3(\varphi) = Span\{e_1, e_2, \cos(\varphi)e_3 + \sin(\varphi)e_7\}.$$

The Lie algebra $\mathfrak{h}(\varphi)$ of the isotropy group of $E^3(\varphi)$ is the nine-dimensional subalgebra of $\mathfrak{so}(7)$ given by the equations

$$\omega_{14} = \omega_{15} = \omega_{16} = \omega_{24} = \omega_{25} = \omega_{26} = 0, \quad \omega_{37} = 0,$$

$$\cos(\varphi)\omega_{34} + \sin(\varphi)\omega_{74} = \cos(\varphi)\omega_{35} + \sin(\varphi)\omega_{75} = \cos(\varphi)\omega_{36} + \sin(\varphi)\omega_{76} = 0,$$

$$\sin(\varphi)\omega_{13} - \cos(\varphi)\omega_{17} = \sin(\varphi)\omega_{23} - \cos(\varphi)\omega_{27} = 0.$$

We calculate the intersection of the Lie algebras $\mathfrak{g}_2$ and $\mathfrak{h}(\varphi)$. It turns out that

$$\dim[\mathfrak{g}_2 \cap \mathfrak{h}(\varphi)] = \begin{cases} 3 & \text{if } \cos(\varphi) \neq 0, \\ 6 & \text{if } \cos(\varphi) = 0. \end{cases}$$

Consequently, the G_2-orbit of the space $E^3(\varphi)$ has dimension 11 (in case $\cos(\varphi) \neq 0$), or dimension eight (in case $\cos(\varphi) = 0$). Moreover, we calculate the value $\omega^3(E^3(\varphi))$:

$$\omega^3(E^3(\varphi)) = \sin(\varphi). \qquad \square$$

Remark 2.7. A three-dimensional subspace $E^3 \subset \mathbb{R}^7$ is said to be G_2-special if its G_2-orbit is an exceptional orbit. The following conditions are equivalent:

(i) E^3 is a special G_2-subspace;
(ii) $|\omega^3(E^3)| = 1$;
(iii) for any vectors $X, Y \in E^3$ and $Z \perp E^3$ the relation $\omega^3(X, Y, Z) = 0$ holds.

3. Topological and geometrical G_2-reductions

Let M^7 be a seven-dimensional manifold and $R(M^7)$ the frame bundle of M^7. We define the bundle $\Lambda^3_+(M^7)$ by

$$\Lambda^3_+(M^7) := R(M^7) \times_{GL(7)} \Lambda^3_+(\mathbb{R}^7) \subset R(M^7) \times_{GL(7)} \Lambda^3(\mathbb{R}^7) = \Lambda^3(M^7).$$

Definition 3.1. A topological G_2-structure on M^7 is a G_2-reduction of the frame bundle $R(M^7)$, i.e. a subbundle P_{G_2} satisfying

$$\begin{array}{ccc} G_2 & \hookrightarrow & GL(7) \\ \downarrow & & \downarrow \\ P_{G_2} & \hookrightarrow & R(M^7) \\ & \searrow \quad \swarrow & \\ & M^7 . & \end{array}$$

Similarly we define topological $SU(2)$-, $SU(3)$- and $Spin(7)$-structures.

The fact that G_2 is a subset of $SO(7)$ and of $Spin(7)$ implies that a G_2-structure P_{G_2} on M^7 induces an orientation of M^7 (i.e. $\omega_1 = 0$), a Riemannian metric g on M^7 such that the corresponding $SO(7)$-bundle equals $P_{G_2} \times_{G_2} SO(7)$, and a spin structure $P_{G_2} \times_{G_2} Spin(7)$ (i.e. $\omega_2 = 0$). Furthermore it defines the following nowhere vanishing spinor $\psi \in \Gamma(S)$ in the real spinor bundle $S = P_{G_2} \times_{G_2} \Delta_7$ of M^7. Since $G_2 \subset Spin(7)$ is the isotropy group of $\psi_0 \in \Delta_7$ the map $\psi : P_{G_2} \longrightarrow \Delta_7$, $\psi(p) = \psi_0$, has the property $\psi(pg) = g^{-1}\psi$ for all $g \in G_2$ and is therefore a section in S. Because of the G_2-invariance of ω_0 the G_2-structure defines in the same way a section ω^3 in $\Lambda^3_+(M^7) = R(M^7) \times_{GL(7)} \Lambda^3_+(\mathbb{R}^7) = P_{G_2} \times_{G_2} \Lambda^3_+(\mathbb{R}^7)$, by $\omega^3 : P_{G_2} \longrightarrow \Lambda^3_+(\mathbb{R}^7)$, $\omega^3(p) = \omega^3_0$. On the other hand the spinor ψ defines a $(2, 1)$-tensor field $A = A_\psi$ (see Eq. (2)) on M^7 and we have $\omega^3 = g(\cdot, A(\cdot, \cdot))$.

Proposition 3.2. *Let M^7 be a compact seven-dimensional manifold. The following conditions are equivalent:*

(i) *M^7 admits a topological $SU(2)$-structure;*
(ii) *M^7 admits a topological $SU(3)$-structure;*
(iii) *M^7 admits a topological G_2-structure;*
(iv) *M^7 admits a topological $Spin(7)$-structure;*
(v) *the first and the second Stiefel–Whitney class of M^7 vanish, i.e. $\omega_1 = 0$ and $\omega_2 = 0$.*

Proof. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and the equivalence (iv) $\Leftrightarrow$ (v) are obvious. It remains to show that the existence of a topological $Spin(7)$-structure implies the existence of a topological $SU(2)$-structure. Let S be the real spinor bundle associated to the given $Spin(7)$-structure. Its dimension equals 8, the dimension of M^7 equals 7. Thus, there exists a section ψ of length one in $\Gamma(S)$. On the other hand, any seven-dimensional orientable compact manifold admits two linearly independent vector fields [26]. Denote these vector fields by X and Y. Then ψ, $X\psi$ and $Y\psi$ are spinor fields, which are linearly independent in any point of M^7. Thus M^7 admits a triple (ψ_1, ψ_2, ψ_3) of spinor fields which are orthogonal in any point. The spinors ψ_i $(i = 1, 2, 3)$ are maps $\psi_i : P_{Spin} \longrightarrow \Delta_7$ satisfying $\psi_i(pg) = g^{-1}\psi_i(p)$ for all $g \in Spin(7)$. Now we can define a $SU(2)$-structure on M^7 by

$$P_{SU(2)} := \{p \in P_{Spin} \mid \psi_1(p) = {}^t(1, 0, \ldots, 0),\ \psi_2(p) = {}^t(0, 1, 0, \ldots, 0),$$
$$\psi_3(p) = {}^t(0, 0, 1, 0, \ldots, 0)\}. \qquad \square$$

Obviously the above-mentioned map from the set of G_2-reductions of $R(M^7)$ into the set of 3-forms is injective. Thus we obtain:

Proposition 3.3. *There is a one-to-one correspondence between the G_2-stuctures on M^7 and the sections of $\Lambda^3_+(M^7)$.*

Similarly we have:

Proposition 3.4. *There is a one-to-one correspondence between the G_2-stuctures on M^7 and the 4-tupels (O, g, P_{Spin}, ψ), where O is an orientation, g a metric, P_{Spin} a spin structure and ψ a spinor field of length one on M^7.*

Now we turn to geometrical G_2-stuctures.

Definition 3.5. Let $P_{G_2} \subset R(M^7)$ be a G_2-reduction and g the associated Riemannian metric. We denote by ∇ the Levi-Civita connection of g. P_{G_2} is said to be geometrical if one of the following equivalent conditions is satisfied:

(i) ∇ reduces to P_{G_2};
(ii) the holonomy group $\mathrm{Hol}(M^7, g)$ of M^7 is contained in G_2;
(iii) the associated 3-form ω^3 is parallel, i.e. $\nabla\omega^3 = 0$;
(iv) the associated spinor field ψ is parallel, i.e. $\nabla\psi = 0$ where, here ∇ is the induced covariant derivative on the spinor bundle S.

An immediate consequence is the following fact proved by Bonan in 1966 (see [5]).

Proposition 3.6. *If g is the Riemannian metric of a geometrical G_2-structure on M^7, then (M^7, g) is Ricci-flat, i.e. $Ric = 0$.*

Proof. Let ψ be the associated section of the spinor bundle S of M^7. Because of $\nabla\psi = 0$ we obtain for the curvature tensor $\Re^S$ of the induced connection ∇ on S

$$\Re^S(X, Y)\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi = 0$$

for all vector fields X, Y on M^7. We recall that the Ricci tensor on M^7 satisfies

$$Ric(X)\varphi = -2\sum_{k=1}^{7} s_k\Re^S(X, s_k)\varphi$$

for any vector field X and any spinor φ on M^7, where $s_1, \dots, s_7$ is a local orthonormal frame (see [2]). Consequently, $Ric(X)\psi = 0$ for all vector fields X and the assertion follows since ψ vanishes nowhere. □

Now we can generalize the condition $\nabla\psi = 0$ and obtain the notion of a nearly parallel G_2-structure.

Definition 3.7. A topological G_2-structure on M^7 is said to be nearly parallel if the associated spinor ψ is a Killing spinor, i.e. there exists a real number λ such that ψ satisfies the differential equation $\nabla_X \psi = \lambda X \psi$ with respect to the Levi-Civita connection of the induced metric.

Differentiating the equation that defines the (2, 1)-tensor A we obtain the following equivalent condition.

Proposition 3.8. *A topological G_2-structure on M^7 is nearly parallel if and only if the associated tensor A satisfies*

$$(\nabla_Z A)(Y, X) = 2\lambda\{g(Y, Z)X - g(X, Z)Y + A(Z, A(Y, X))\} \tag{5}$$

with respect to the Levi-Civita connection of the induced metric where λ is the same number as in Definition 3.7.

Now we translate this condition into a differential equation for the 3-form ω^3.

Proposition 3.9. *A topological G_2-structure on M^7 is nearly parallel if and only if the associated 3-form ω^3 satisfies*

$$\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3)$$

with respect to the Levi-Civita connection of the induced metric where λ is the same number as in Definition 3.7.

Proof. The 3-form ω^3 is defined by $\omega^3(X, Y, Z) = g(X, A(Y, Z))$. Differentiating this equation we observe that Eq. (5) is equivalent to

$$(\nabla_Z \omega^3)(W, Y, X) = 2\lambda g(Z, g(W, X)Y - g(W, Y)X - A(W, A(Y, X)))$$

for any vector field Z. For fixed Z the 3-form on the right-hand side of this equation equals locally

$$\begin{aligned} &2\lambda \sum_{i<j<k} g(Z, g(s_i, s_k)s_j - g(s_i, s_j)e_k - A(s_i, A(s_j, s_k)))s_i \wedge s_j \wedge s_k \\ &\quad = -2\lambda \sum_{i<j<k} g(Z, A(s_i, A(s_j, s_k)))s_i \wedge s_j \wedge s_k, \end{aligned}$$

where $s_1, \ldots, s_7$ is a section of the G_2-structure on M^7. However, we obtain from

$$\begin{aligned} \omega^3 = {} & s_1 \wedge s_2 \wedge s_7 + s_1 \wedge s_3 \wedge s_5 - s_1 \wedge s_4 \wedge s_6 \\ & - s_2 \wedge s_3 \wedge s_6 - s_2 \wedge s_4 \wedge s_5 + s_3 \wedge s_4 \wedge s_7 + s_5 \wedge s_6 \wedge s_7, \end{aligned}$$

on the one hand all $A(s_i, A(s_j, s_k))$, and on the other hand $*\omega^3$. The assertion follows by comparing these terms. □

In the same way as in the case of geometrical G_2-structures we prove:

Proposition 3.10. *If g is the Riemannian metric of a nearly parallel G_2-structure on M^7, then (M^7, g) is an Einstein space.*

Proof. The induced spinor ψ is a Killing spinor and we obtain from $\nabla_X \psi = \lambda X \cdot \psi$

$$\Re^S(X,Y)\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi = 2\lambda^2(Y \cdot X + g(X,Y)) \cdot \psi.$$

This yields for the Ricci tensor

$$Ric(X)\psi = -2\sum_{k=1}^{7} s_k \Re^S(X, s_k)\psi = -4\lambda^2 \sum_{k=1}^{7} s_k(s_k X + g(X, s_k))\psi = 24\lambda^2 X\psi.$$

Since ψ has no zeros, $Ric(X) = 24\lambda^2 X$ and, therefore, (M^7, g) is an Einstein space of constant scalar curvature $R = 7 \cdot 24\lambda^2$. □

Next we generalize the following result of Gray and Fernandez.

Proposition 3.11 [9,20,21]. *Let $P_{G_2} \subset R(M^7)$ be a topological G_2-reduction, g its induced metric, ω^3 the induced 3-form and $*$ the Hodge operator. Then the following conditions are equivalent:*
(i) *P_{G_2} is geometrical;*
(ii) $\nabla\omega^3 = 0$;
(iii) $\mathrm{d}\omega^3 = 0$, $\mathrm{d} * \omega^3 = 0$.

We transfer the proof of this proposition given in [9] to the case of nearly parallel G_2-reductions and obtain:

Proposition 3.12. *Let $P_{G_2} \subset R(M^7)$ be a topological G_2-reduction, g its induced metric, ω^3 the induced 3-form, ψ the induced spinor and $*$ the Hodge operator. Then the following conditions are equivalent:*
(i) *P_{G_2} is nearly parallel, i.e. the spinor ψ satisfies $\nabla_X\psi = \lambda X\psi$;*
(ii) $\nabla_Z\omega^3 = -2\lambda(Z \lrcorner *\omega^3)$;
(iii) $\delta\omega^3 = 0$, $\mathrm{d}\omega^3 = -8\lambda * \omega^3$.

Proof. The 3-form ω^3 defines the metric g. Let $\Sigma_g \subset \Lambda^3_+(M^7)$ be the set of 3-forms that define this metric also. The fibre of Σ_g equals the $SO(7)$-orbit of ω^3, i.e. $SO(7)/G_2$. Its tangent space $T(SO(7)/G_2)$ is G_2-invariant and seven-dimensional, therefore $T(SO(7)/G_2) = S_{\omega^3} := \{X \lrcorner *\omega^3 \mid X \in TM^7\}$. Since ω^3 is a section in Σ_g and ∇ is a covariant derivative in Σ_g, the covariant derivative $\nabla\omega^3$ is a section of $T^*M^7 \otimes S_{\omega^3}$. We consider now the projection p_1 defined by

$$p_1 : T^*M^7 \otimes S_{\omega^3} \ni \alpha^1 \otimes \alpha^3 \longmapsto \alpha^1 \wedge \alpha^3 \in \Lambda^4$$

and the contraction

$$p_2 : T^*M^7 \otimes S_{\omega^3} \longrightarrow \Lambda^2.$$

By comparing the decomposition of $T^*M^7 \otimes S_{\omega^3}$ and $\Lambda^4 \oplus \Lambda^2$ into irreducible G_2-subspaces we see that the sum of p_1 and p_2

$$p_1 \oplus p_2 : T^*M^7 \otimes S_{\omega^3} \longrightarrow \Lambda^4 \oplus \Lambda^2$$

is injective. Consequently, $\nabla_X \psi = \lambda X \psi$ is equivalent to

$$p_1(\nabla\omega^3) = -2\lambda p_1(\cdot \lrcorner *\omega^3) = -2\lambda \sum_{i=1}^{7} s_i \wedge s_i \lrcorner *\omega^3 = -8\lambda * \omega^3,$$

$$p_2(\nabla\omega^3) = -2\lambda p_2(\cdot \lrcorner *\omega^3) = -2\lambda \sum_{i=1}^{7} *\omega^3(s_i, s_i, \ldots) = 0.$$

The assertion now follows from $p_1(\nabla\omega^3) = \mathrm{d}\omega^3$, $p_2(\nabla\omega^3) = \delta\omega^3$. □

Remark 3.13. There is the following difference between the cases $\lambda = 0$ and $\lambda \neq 0$: we proved that a G_2-structure is nearly parallel if and only if for the induced 3-form ω^3 the equations $\delta\omega^3 = 0$, $\mathrm{d}\omega^3 = -8\lambda * \omega^3$ hold. In case $\lambda = 0$, the resulting equations $\mathrm{d}\omega^3 = 0$ and $\delta\omega^3 = 0$ are independent. In case $\lambda \neq 0$, the condition $\mathrm{d}\omega^3 = -8\lambda * \omega^3$ implies $\delta\omega^3 = 0$.

4. Nearly parallel G_2-structures, Killing spinors and contact geometry

We summarize now several results on nearly parallel G_2-structures. A general reference is the book [2]. In particular, we derive necessary geometric conditions for the underlying Riemannian metric and we introduce three types of nearly parallel G_2-structures depending on the number of Killing spinors. Finally we discuss the compact examples of each type known up to now.

Let (M^7, g) be a compact Riemannian spin manifold with a Killing spinor ψ,

$$\nabla_X \psi = \lambda X \cdot \psi,$$

and denote by ω^3 the corresponding 3-form satisfying the differential equation

$$\mathrm{d}\omega^3 = -8\lambda * \omega^3.$$

Then M^7 is an Einstein manifold of positive scalar curvature $R = 4 \cdot 7 \cdot 6 \cdot \lambda^2 = 168\lambda^2$ and, consequently, the fundamental group $\pi_1(M)$ is finite. In case $\lambda \neq 0$ the Riemannian manifold (M^7, g) is locally irreducible and not locally symmetric except if it has constant sectional curvature (see [2]). Using the associated nearly parallel G_2-structure we decompose the bundles of forms $\Lambda^p(M^7)$ into the irreducible components mentioned above. The curvature tensor

$$\mathfrak{R} : \Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \longrightarrow \Lambda^2_7 \oplus \Lambda^2_{14} = \Lambda^2$$

splits into the scalar curvature and the Weyl tensor W:

$$\mathfrak{R} = W - \tfrac{1}{42} R.$$

The Weyl tensor satisfies several algebraic equations. They can be formulated in the following way. For any 2-form $\omega^2 \in \Lambda^2$ the Clifford product $W(\omega^2) \cdot \psi$ vanishes, i.e.

$$W(\omega^2) \cdot \psi = 0$$

holds (see [2]). Since Λ^2_{14} is the Lie algebra of the group G_2, being the isotropy group of the spinor ψ, we conclude that the Weyl tensor has the form

$$W = \begin{pmatrix} 0 & 0 \\ 0 & W_{14} \end{pmatrix},$$

where $W_{14} : \Lambda^2_{14} \longrightarrow \Lambda^2_{14}$ is a symmetric endomorphism. In case $W_{14} \neq 0$, the holonomy representation $\mathrm{Hol}^0 \longrightarrow SO(7)$ is irreducible and we can apply Berger's Holonomy Theorem. Since $\dim(M^7) = 7$, there are two possibilities: $\mathrm{Hol}^0 = G_2$ or $\mathrm{Hol}^0 = SO(7)$. The case of $\mathrm{Hol}^0 = G_2$ cannot occur since M^7 is an Einstein space with positive scalar curvature ($\lambda \neq 0$). Consequently, the Riemannian manifold M^7 is – at least from the point of view of holonomy theory – of general type: $\mathrm{Hol}^0 = SO(7)$. Since the fundamental group of M^7 is finite we can without loss of generality assume that M^7 is simply connected, $\pi_1(M^7) = 0$. Furthermore, we exclude the case of the space of constant curvature, i.e. $M^7 \neq S^7$. Denote by $KS(M^7, g)$ the space of all Killing spinors,

$$KS(M^7, g) = \{\psi \in \Gamma(S) \colon \nabla_X \psi = \lambda X \cdot \psi \text{ for all vectors } X \in T(M^7)\}.$$

The dimension of $KS(M^7, g)$ is bounded by three, $\dim[KS(M^7, g)] \leq 3$ (see [2]). The nearly parallel G_2-structures split into three different types:

nearly parallel G_2-structures of type 1: $\dim[KS] = 1$ (proper G_2-structures),
nearly parallel G_2-structures of type 2: $\dim[KS] = 2$,
nearly parallel G_2-structures of type 3: $\dim[KS] = 3$.

The nearly parallel G_2-structures of type 2 and 3 are described using the language of contact geometry. In fact, Th. Friedrich and I. Kath observed that a simply connected seven-dimensional Riemannian spin manifold with scalar curvature $R = 42$ admits at least

two Killing spinors $\Longleftrightarrow$ M^7 is an Einstein–Sasakian manifold,

three Killing spinors $\Longleftrightarrow$ M^7 is a 3-Sasakian manifold

(see [16]; for the definition of a Sasakian manifold also see next section). For the G_2-structures of type 1 we also use the notion of a *proper G_2-structure*.

Examples of nearly parallel G_2-structures of type 3 (i.e. 3-Sasakian manifolds) are known (see Table 1). We have the sphere S^7, the space $N(1, 1) = SU(3)/S^1$ and these are the only regular 3-Sasakian manifolds in dimension seven (see [16]). In the past Boyer et al. obtained non-regular examples $S(p_1, p_2, p_3)$ (see [6,7]). Up to now, strong topological conditions for a compact seven-dimensional manifold M^7 in order to admit a 3-Sasakian structure are not known. For example, it seems to be an open question whether the manifold $S^2 \times S^5$ posseses such a structure or not! This special question is interesting since a seven-dimensional

Table 1
Examples of nearly parallel G_2-structures of type 3 (3-Sasakian manifolds)

M^7	$\mathrm{Iso}_o(M^7)$	dim[Iso]
$N(1,1)$	$SU(3) \times SU(2)$	11
$S(p_1, p_2, p_3)$	depends on p_i	< 8

Table 2
Examples of nearly parallel G_2-structures of type 2 (Einstein–Sasakian manifolds)

X^6	M^7	$\mathrm{Iso}_o(M^7)$	dim[Iso]
$F(1,2)$	$N(1,1)$	$SU(3) \times SU(2)$	11
$S^2 \times S^2 \times S^2$	$Q(1,1,1)$	$SU(2) \times SU(2) \times SU(2) \times U(1)$	10
$\mathbb{C}\mathrm{P}^2 \times S^2$	$M(3,2)$	$SU(3) \times SU(2) \times U(1)$	12
$G_{5,2}$	$V_{5,2}$	$SO(5) \times U(1)$	11
$P_k \times S^2$	M^7_k $(3 \leq k \leq 8)$	$SO(3) \times U(1)$	4

manifold with 3-Sasakian structure and being the product of two lower-dimensional manifolds must be diffeomorphic to $S^2 \times S^5$.

Nearly parallel G_2-structures of type 2 (i.e. Einstein–Sasakian manifolds) can be obtained as principal S^1-bundles over six-dimensional Kähler–Einstein manifolds with positive scalar curvature. Indeed, let X^6 be a Kähler–Einstein manifold with positive scalar curvature and denote by $c_1(X^6)$ its first Chern class. Let $A > 0$ be the largest integer such that $c_1(X^6)/A$ is an integral cohomology class. Consider the principal S^1-bundle $S^1 \longrightarrow M^7 \longrightarrow X^6$ with Chern class $c_1^* = c_1(X^6)/A$. Then M^7 is simply connected and admits an Einstein–Sasakian structure. Using the described construction we obtain the regular Einstein–Sasakian manifolds presented in Table 2, where P_k $(3 \leq k \leq 8)$ denotes one of the del Pezzo surfaces with a Kähler–Einstein metric of positive scalar curvature. The spaces $N(1,1)$, $Q(1,1,1)$, $M(3,2)$ and the Stiefel manifold $V_{5,2}$ are homogeneous spaces together with some invariant Einstein metric. Table 2 contains also the isometry group of the Einstein–Sasakian manifold M^7 as well as its dimension (see [11]).

There are three examples of nearly parallel G_2-structures of type 1, i.e. proper G_2-structures (see Table 3). The first example is the so-called squashed 7-sphere. Indeed, the standard sphere (S^7, g_{can}) is a Riemannian submersion over the projective space $\mathbb{H}\mathrm{P}^1$ with fibre S^3. Scaling the canonical metric in the fibre S^3, there exists a second scaling factor such that the metric g_1 on S^7 is an Einstein metric. It turns out that (S^7, g_1) admits exactly one Killing spinor. The second example is the homogeneous space $N(k,l) = SU(3)/S^1_{k,l}$ where the embedding of the group $S^1 = U(1)$ into $SU(3)$ is given by

$$S^1 \ni z \longmapsto \mathit{diag}(z^k, z^l, z^{-(k+l)}) \in SU(3).$$

These spaces have two homogeneous Einstein metrics. In case $(k,l) = (1,1)$ one of these Einstein metrics is the 3-Sasakian structure mentioned above and the second Einstein metric admits one Killing spinor. In case $(k,l) \neq (1,1)$, there exists only one Killing spinor for

Table 3
Examples of nearly parallel G_2-structures of type 1

M^7	$\mathrm{Iso}_o(M^7)$	dim [Iso]
$(S^7, g_{\mathrm{squas}})$	$Sp(2) \times Sp(1)$	13
$N(k,l), (k,l) \neq (1,1)$	$SU(3) \times U(1)$	9
$SO(5)/SO(3)$	$SO(5)$	10

each of these two metrics, i.e. the nearly parallel G_2-structure is of type 1 (a proper G_2-structure). The third example is a special Riemannian metric on $SO(5)/SO(3)$ with one Killing spinor (see [8]). The isotropy representation of this space is the unique seven-dimensional irreducible representation of the group $SO(3) \longrightarrow G_2 \subset SO(7)$.

Remark 4.1. As we mentioned before, strong topological obstructions for the existence of a 3-Sasakian metric on a compact seven-dimensional spin manifold are not known (very recently, the obstruction $b_3(M^7) = 0$ was found, see [17]). The same situation happens in case of an Einstein–Sasakian metric with positive scalar curvature. This gives rise to the following question:

Do there exist compact, simply connected spin manifolds M^7 with a nearly parallel G_2-structure of type 1 (resp. 2) which cannot admit – for example for topological reasons – any Einstein–Sasakian (resp. 3-Sasakian) metric at all?

5. New examples

In this section we construct new examples of nearly parallel G_2-structures and show that they are of type 1, i.e. they are proper G_2-structures. Let us recall the definition of a Sasakian structure.

Definition 5.1. A vector field V on a Riemannian manifold (M, g) is called a *Sasakian structure* if the following conditions are satisfied:
(1) V is a Killing vector field of unit length;
(2) the (1, 1)-tensor φ defined by $\varphi = -\nabla V$ is an almost complex structure on the distribution orthogonal to V ($\varphi^2 = -1$ and $\varphi = -\varphi^*$ on $V^\perp$);
(3) $(\nabla_X \varphi)Y = g(X, Y)\, V - g(V, Y)\, X$ for all vectors X, Y.

Definition 5.2. A triple (V_1, V_2, V_3) is called a *3-Sasakian structure* on M if the following conditions are satisfied:
(1) the vector V_i defines a Sasakian structure for each $i = 1, 2, 3$;
(2) the frame (V_1, V_2, V_3) is orthonormal;
(3) for each permutation (i, j, k) of signature δ, we have $\nabla_{V_i} V_j = (-1)^\delta\ V_k$;
(4) on the distribution orthogonal to (V_1, V_2, V_3), the tensors $\varphi_i = -\nabla V_i$ satisfy $\varphi_i\, \varphi_j = (-1)^\delta\, \varphi_k$.

Consider a Riemannian manifold M^7 of dimension seven, admitting a 3-Sasakian structure. A vector is called *horizontal* if it is orthogonal to each V_i and *vertical* if it is a linear combination of V_i. Define, for $s > 0$, the metric g^s on M^7 by $g^s(X, Y) = g(X, Y)$ if X (or Y) is horizontal, and $g^s(V, W) = s^2 g(V, W)$ for vertical V, W.

A straightforward computation gives the following:

Lemma 5.3. *The manifold (M^7, g^s) is Einstein if and only if $s = 1$ or $s = 1/\sqrt{5}$.*

The 3-Sasakian manifold (M^7, g^1) admits, by definition, a nearly parallel G_2-structure of type 3. On the other hand, by Proposition 3.10, every nearly parallel G_2-structure on M defines an Einstein metric. Hence, the manifold (M^7, g^s) with $s = 1/\sqrt{5}$ is a natural candidate for a nearly parallel G_2-structure. Indeed, we have:

Theorem 5.4. *The manifold (M^7, g^s) admits a nearly parallel G_2-structure for $s = 1/\sqrt{5}$.*

Proof. Fix $s > 0$ and a local orthonormal frame $X_1, \ldots, X_4$ of the horizontal distribution. Let Z_a $(a = 1, 2, 3)$ be the vector $Z_a := V_a/s$ and denote by ∇ the Levi-Civita connection of the metric $g = g^1$. We define a 3-form ω by

$$\omega := F_1 + F_2,$$

where $F_1 := Z_1 \wedge Z_2 \wedge Z_3$, $F_2 := \sum_a Z_a \wedge \omega_a$ and $\omega_a := \frac{1}{2}\sum_i X_i \wedge \nabla_{X_i} V_a$.

The 3-form ω is clearly in $\Lambda^3_+(M^7)$. Denote by $*$ the Hodge operator with respect to the metric g^s. Then we calculate the forms $*F_1$ and $*F_2$:

$$6 * F_1 = \sum_a \omega_a \wedge \omega_a, \qquad *F_2 = Z_1 \wedge Z_2 \wedge \omega_3 + Z_2 \wedge Z_3 \wedge \omega_1 + Z_3 \wedge Z_1 \wedge \omega_2.$$

A straightforward computation yields the formulas

$$\begin{aligned} \mathrm{d}Z_1 &= 2s\omega_1 - (2/s) Z_2 \wedge Z_3, & \mathrm{d}Z_2 &= 2s\omega_2 - (2/s) Z_3 \wedge Z_1, \\ \mathrm{d}Z_3 &= 2s\omega_3 - (2/s) Z_1 \wedge Z_2, & \mathrm{d}F_1 &= \mathrm{d}(Z_1 \wedge Z_2 \wedge Z_3) = 2s(*F_2). \end{aligned}$$

Now we compute

$$\begin{aligned} \mathrm{d}\omega_1 &= \mathrm{d}\left((1/2s)\,\mathrm{d}Z_1 + (1/s^2) Z_2 \wedge Z_3\right) \\ &= (1/s^2)(\mathrm{d}Z_2 \wedge Z_3 - Z_2 \wedge \mathrm{d}Z_3) \\ &= (2/s)(\omega_2 \wedge Z_3 - \omega_3 \wedge Z_2), \\ \mathrm{d}\omega_2 &= (2/s)(\omega_3 \wedge Z_1 - \omega_1 \wedge Z_3), \qquad \mathrm{d}\omega_3 = (2/s)(\omega_1 \wedge Z_2 - \omega_2 \wedge Z_1), \\ \mathrm{d}F_2 &= \sum_a \mathrm{d}Z_a \wedge \omega_a - \sum_a Z_a \wedge \mathrm{d}\omega_a = 12s(*F_1) + (2/s)(*F_2). \end{aligned}$$

Finally we obtain $\mathrm{d}\omega = \mathrm{d}(F_1 + F_2) = 12s(*F_1) + (2s + (2/s))(*F_2)$. So $\mathrm{d}\omega$ is a scalar multiple of $*\omega$ if and only if $12s = 2s + 2/s$. □

As remarked in Section 4, the only known examples of proper nearly parallel G_2-structures – up to now – are the squashed 7-sphere, the Wallach spaces $N(k, l)$ and an

Einstein metric on $SO(5)/SO(3)$ related to the irreducible representation $SO(3) \to G_2 \to SO(7)$. The importance of Theorem 5.4 can thus be seen in the light of the following result:

Theorem 5.5. *The nearly parallel G_2-structures constructed in Theorem 5.4 are proper.*

Proof. Suppose that a constant multiple k of the metric g^s on M^7 admits an Einstein–Sasakian structure given by the Killing vector field ξ. Denote by $\mathfrak{R}$ the curvature tensor of (M^7, g) and by $\mathfrak{R}^0$ the curvature tensor of (M^7, kg^s). Then we obtain from Lemma 4 of [2, p. 78]:

$$g^s(\mathfrak{R}^0(X, Y)\xi, V_a) = k[g^s(Y, \xi)g^s(X, V_a) - g^s(X, \xi)g^s(Y, V_a)]. \tag{$*$}$$

Choosing X and Y horizontal we obtain

$$g^s(\mathfrak{R}^0(X, Y)\xi, V_a) = 0.$$

On the other hand, comparing the Levi-Civita connection ∇ of the metric g with the Levi-Civita connection ∇^0 of the metric g^s we calculate

$$\mathfrak{R}^0(X, Y)V_a = s^2\mathfrak{R}(X, Y)V_a + (s^2 - 1)\nabla_{[X,Y]^V} V_a = (s^2 - 1)\nabla_{[X,Y]^V} V_a.$$

Here we apply the same lemma for V_a as Sasakian structure on (M^7, g). Consequently, ξ is perpendicular to all vectors of the form $\nabla_{[X,Y]^V} V_a$. It is easy to see that the set of all these vectors is just the vertical distribution, so ξ is horizontal.

Next, taking $X = V_1, a = 2$ and Y horizontal in Eq. ($*$), one obtains

$$g^s(\mathfrak{R}^0(V_1, Y)\xi, V_2) = 0. \tag{$**$}$$

The vector $[Y, V_1]$ is a horizontal one and we can calculate

$$\begin{aligned}\mathfrak{R}^0(V_1, Y)V_2 &= \nabla^0_{V_1}\nabla^0_Y V_2 - \nabla^0_Y\nabla^0_{V_1} V_2 - \nabla^0_{[V_1,Y]} V_2\\ &= s^2(\nabla^0_{V_1}\nabla_Y V_2 - \nabla_{V_1}\nabla_Y V_2) + s^2\mathfrak{R}(V_1, Y)V_2\\ &= s^2(\nabla^0_{V_1}\nabla_Y V_2 - \nabla_{V_1}\nabla_Y V_2)\end{aligned}$$

by similar arguments. Now $\nabla_Y V_2$ runs through all horizontal vector fields when Y is horizontal. Together with ($**$) we obtain that ξ is perpendicular to all vectors of the form $\nabla^0_{V_1} Z - \nabla_{V_1} Z$. The relation

$$\nabla^0_{V_1} Z - \nabla_{V_1} Z = (\nabla^0_{V_1} Z - [Z, V_1]) - (\nabla_{V_1} Z - [Z, V_1]) = (s^2 - 1)\nabla_Z V_1$$

shows that ξ is also perpendicular to all horizontal vectors, a contradiction. □

Our new examples of nearly parallel G_2-structures are all proper. The recent work of Boyer et al. [7] provides a multitude of new examples of strongly inhomogeneous 7-manifolds admitting a 3-Sasakian structure. By our previous theorems, they generate the first examples of strongly inhomogeneous proper nearly parallel G_2-structures. However, these examples arise from a deformation of the 3-Sasakian structure and therefore they live on manifolds with 3-Sasakian metric.

As K. Galicki pointed out to us, he also proved the result of Theorem 5.4 in a joint paper with S. Salamon (in preparation).

6. The automorphism group of a nearly parallel G_2-structure

We consider a compact, seven-dimensional manifold M^7 with a nearly parallel G_2-structure and denote by ω^3 its 3-form. Then we have the differential equations

$$\nabla_X\omega^3 = -2\lambda(X \lrcorner *\omega^3),\ \ \mathrm{d}\omega^3 = -8\lambda * \omega^3,\ \ \lambda \neq 0.$$

Let X be a vector field preserving the 3-form, i.e.

$$L_X\omega^3 = \mathrm{d}(X \lrcorner \omega^3) + X \lrcorner \mathrm{d}\omega^3 = \mathrm{d}(X \lrcorner \omega^3) - 8\lambda(X \lrcorner *\omega^3) = 0.$$

In particular, X is a Killing vector field of the Riemannian metric g and $\nabla X \in \Gamma(T \otimes T)$ is anti-symmetric and coincides – up to a multiple – with the exterior derivative of the 1-form X:

$$\nabla X = \tfrac{1}{2}\,\mathrm{d}X.$$

We now calculate the form $\mathrm{d}(X \lrcorner \omega^3)$ using the differential equation for ω^3:

$$\begin{aligned}\mathrm{d}(X \lrcorner \omega^3)(\alpha,\beta,\gamma) &= (\nabla_\alpha\omega^3)(X,\beta,\gamma) - (\nabla_\beta\omega^3)(X,\alpha,\gamma) + (\nabla_\gamma\omega^3)(X,\alpha,\beta)\\ &\quad + \omega^3(\nabla_\alpha X,\beta,\gamma) - \omega^3(\nabla_\beta X,\alpha,\gamma) + \omega^3(\nabla_\gamma X,\alpha,\beta)\\ &= 6\lambda(*\omega^3)(X,\alpha,\beta,\gamma) + \omega^3(\nabla_\alpha X,\beta,\gamma)\\ &\quad - \omega^3(\nabla_\beta X;\alpha,\gamma) + \omega^3(\nabla_\gamma X,\alpha,\beta).\end{aligned}$$

The equation $\mathrm{d}(X \lrcorner \omega^3) - 8\lambda(X \lrcorner *\omega^3) = 0$ becomes:

$$\begin{aligned}&2\lambda(X \lrcorner *\omega^3)(\alpha,\beta,\gamma)\\ &\quad = \omega^3(\nabla_\alpha X,\beta,\gamma) - \omega^3(\nabla_\beta X,\alpha,\gamma) + \omega^3(\nabla_\gamma X,\alpha,\beta)\\ &\quad = \tfrac{1}{2}\{\omega^3(\alpha \lrcorner \mathrm{d}X,\beta,\gamma) - \omega^3(\beta \lrcorner \mathrm{d}X,\alpha,\gamma) + \omega^3(\gamma \lrcorner \mathrm{d}X,\alpha,\beta)\}.\end{aligned}$$

We apply now the following easy algebraic observation:

Lemma 6.1. *Let η^2 be a 2-form and denote by $\pi_7(\eta^2)$ its Λ^2_7-component with respect to the decomposition $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$. Suppose that $\pi_7(\eta^2)$ is given by a vector Z, i.e. $\pi_7(\eta^2) = Z \lrcorner \omega^3$. Then*

$$\begin{aligned}&\omega^3(\alpha \lrcorner \eta^2,\beta,\gamma) - \omega^3(\beta \lrcorner \eta^2,\alpha,\gamma) + \omega^3(\gamma \lrcorner \eta^2,\alpha,\beta)\\ &\quad = 3(Z \lrcorner *\omega^3)(\alpha,\beta,\gamma).\end{aligned}$$

The condition that the vector field X preserves the 3-form becomes equivalent to

$$2\lambda(X \lrcorner *\omega^3) = \tfrac{1}{2} \cdot 3(Z \lrcorner *\omega^3),$$

where $\pi_7(\mathrm{d}X) = Z \lrcorner \omega^3$. This implies $Z = \frac{4}{3} \cdot \lambda \cdot X$ and consequently we have proved:

Theorem 6.2. *A Killing vector field X preserves a nearly parallel G_2-structure ω^3 if and only if*

$$\pi_7(\mathrm{d}X) = \tfrac{4}{3} \cdot \lambda \cdot (X \lrcorner \omega^3).$$

We now use Stokes Theorem as well as the identities given in Propositions 2.4 and 2.5 in order to obtain the following relation between the L^2-norms $|\cdot|$ of X and the Λ^2_{14}-part $\pi_{14}(\mathrm{d}X)$ of $\mathrm{d}X$:

Theorem 6.3. *Let X be a Killing vector field preserving a nearly parallel G_2-structure ω^3 on a closed manifold M^7. Then*

$$\tfrac{128}{3}\lambda^2|X|^2 = |\pi_{14}(\mathrm{d}X)|^2.$$

Proof. We start with the algebraic identity $\mathrm{d}X \wedge \mathrm{d}X \wedge \omega^3 = 2|\pi_7(\mathrm{d}X)|^2 - |\pi_{14}(\mathrm{d}X)|^2$ which is valid for any 2-form $\eta(= \mathrm{d}X)$. By Stokes Theorem and Propositions 2.4 and 2.5 we obtain

$$\begin{aligned}\int \mathrm{d}X \wedge \mathrm{d}X \wedge \omega^3 &= \int X \wedge \mathrm{d}X \wedge \mathrm{d}\omega^3 = -8\lambda \int X \wedge \mathrm{d}X \wedge (*\omega^3)\\ &= -8\lambda \int X \wedge \pi_7(\mathrm{d}X) \wedge (*\omega^3)\\ &= -\frac{32}{3}\lambda^2 \int X \wedge (X \lrcorner \omega^3) \wedge (*\omega^2)\\ &= -32\lambda^2 \int X \wedge (*X) = -32\lambda^2|X|^2.\end{aligned}$$

Therefore we get

$$2|\pi_7(\mathrm{d}X)|^2 - |\pi_{14}(\mathrm{d}X)|^2 = -32\lambda^2|X|^2.$$

Using the equation, $\pi_7(\mathrm{d}X) = \frac{4}{3} \cdot \lambda \cdot (X \lrcorner \omega^3)$, we have

$$|\pi_7(\mathrm{d}X)|^2 = \tfrac{16}{3}\lambda^2|X|^2,$$

and the formula follows immediately. □

Consider a component $\Sigma \subset M^7$ of the zero set of X. Since X is a Killing vector field, Σ is a totally geodesic submanifold of even codimension. Suppose that $\dim[\Sigma] = 5$. Then at any point of Σ we obtain that $0 \neq \mathrm{d}X \in \Lambda^2_{14}$ has rank 2 ($\pi_7(\mathrm{d}X) = 0$!). This implies $\mathrm{d}X \wedge \mathrm{d}X = 0$. On the other hand, since $\mathrm{d}X \in \Lambda^2_{14}$ we have $\mathrm{d}X \wedge \mathrm{d}X \wedge \omega^3 = -|\mathrm{d}X|^2$ (see the definition of the space Λ^2_{14}), a contradiction. This yields:

Corollary 6.4. *Any connected component of the zero set of a Killing vector field X preserving a nearly parallel G_2-structure ω^3 has dimension one or three.*

We investigate now the geometry of the three-dimensional components of the zero set Σ.

Theorem 6.5. *Let $\Sigma^3 \subset M^7$ be a three-dimensional component of the zero set of a Killing vector field preserving a nearly parallel G_2-structure. Then*
(i) *the tangent spaces $T(\Sigma^3) \subset T(M^7)$ are G_2-special, i.e. the restriction of ω^3 to Σ^3 is the volume form of Σ^3;*
(ii) *Σ^3 is a space form of positive sectional curvature $K = \frac{1}{42}R$.*

Proof. The equation $X \lrcorner \mathrm{d}\omega^3 + \mathrm{d}(X \lrcorner \omega^3) = 0$ yields at any point $m \in \Sigma^3$ and for any three vectors $\alpha, \beta, \gamma \in T_m(M^7)$ the relation

$$0 = \mathrm{d}(X \lrcorner \omega^3)(\alpha, \beta, \gamma) = \omega^3(\nabla_\alpha X, \beta, \gamma) - \omega^3(\nabla_\beta X, \alpha, \gamma) + \omega^3(\nabla_\gamma X, \alpha, \beta).$$

Let $e_1, e_2, \ldots, e_7$ be a local orthonormal frame in the G_2-bundle such that $e_1(m), e_2(m)$ belong to the tangent space $T_m(\Sigma^3)$. There exists a frame with the required property since the group G_2 acts transitively on the Grassmannian manifold $G_2(\mathbb{R}^7)$. With respect to

$$\nabla_{e_1} X = \nabla_{e_2} X = 0$$

we obtain ($\beta = e_1, \gamma = e_2$) $\omega^3(\nabla_\alpha X, e_1, e_2) = 0$. The vectors $\nabla_\alpha X$, $\alpha \in T_m(M^7)$, generate the normal space of $T_m(\Sigma^3)$ and, therefore, the latter equation means that the subspace $T_m(\Sigma^3) \subset T_m(M^7)$ is of special G_2-type (see Proposition 2.6). In particular, the vector e_7 is the third vector tangent to Σ^3 at the point $m \in \Sigma^3$. The 2-forms

$$e_2 \wedge e_7 + e_3 \wedge e_5 - e_4 \wedge e_6, \quad e_1 \wedge e_7 + e_3 \wedge e_6 + e_4 \wedge e_5,$$
$$e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6$$

are elements of Λ^2_7. The curvature tensor $\mathfrak{R}$ of M^7 acts on forms of the type Λ^2_7 by the scalar multiplication by $-\frac{1}{42}R$ (see Section 4). This implies

$$\mathfrak{R}(e_2 \wedge e_7 + e_3 \wedge e_5 - e_4 \wedge e_6) = -\tfrac{1}{42}R(e_2 \wedge e_7 + e_3 \wedge e_5 - e_4 \wedge e_6)$$

and, finally, $R_{2772} = \frac{1}{42}R$ because Σ^3 is a totally geodesic submanifold (i.e. $R_{3527} = R_{4627} = 0$ for example). Similarly we obtain $R_{1771} = R_{1221} = \frac{1}{42}R$ and, hence, Σ^3 is a space form of positive sectional curvature $K = \frac{1}{42}R$. □

Let $H \subset G = Aut(M^7, \omega^3)$ be a subgroup of the connected component G of the automorphism group of a nearly parallel G_2-structure ($\lambda \neq 0$) and suppose that for some point $m^* \in M^7$ the H-orbit $N^4 = H \cdot m^*$ is a four-dimensional submanifold. Then $(*\omega^3)$ is an H-invariant 4-form on N^4, i.e. a constant multiple of the volume form of N^4. On the other hand, we have

$$(-8\lambda) \int_{N^4} (*\omega^3) = \int_{N^4} \mathrm{d}\omega^3 = 0$$

and, thus, $*\omega^3$ vanishes on N^4. This implies that ω^3 vanishes on the normal bundle $T^\perp(N^4)$. Using Proposition 2.6 we obtain a local frame $e_1, e_2, \ldots, e_7$ in the G_2-bundle over N^4 such that e_4, e_5, e_6, e_7 span the tangent space $T(N^4)$ of N^4. Moreover, on N^4 the formula

$$\omega^3|_{N^4} = e_5 \wedge e_6 \wedge e_7$$

holds, i.e. $\omega^3|_{N^4}$ is a 3-form on N^4 of length one. Denote by ξ the tangent vector field on N^4 corresponding to this 3-form under the Hodge operator of N^4 ($\xi = e_4$). Then we have

$$\xi \lrcorner \omega^3 = 0, \quad dN^4 = \xi \wedge \omega^3.$$

The 3-form ω^3 is invariant under the flow of the vector field ξ on N^4:

$$L_\xi(\omega^3) = \xi \lrcorner d\omega^3 + d(\xi \lrcorner \omega^3) = -8\lambda(\xi \lrcorner *\omega^3) + 0 = 0.$$

We summarize the result in the following:

Theorem 6.6. *Let $N^4 = H \cdot m^*$ be a four-dimensional orbit, $H \subset G = Aut(M^7, \omega^3)$. Then the restriction of ω^3 to N^4 is a 3-form on N^4 with length one. Moreover, there exists a vector field ξ such that*
(i) $\xi \lrcorner \omega^3 = 0$, $dN^4 = \xi \wedge \omega^3$;
(ii) $L_\xi(\omega^3) = 0$.
In particular, the Euler characteristic $\chi(N^4)$ of N^4 vanishes, $\chi(N^4) = 0$.

Corollary 6.7. *The isotropy representation $G(m) \longrightarrow GL(T_m(M^7))$ at any point $m \in N^4$ of a four-dimensional orbit N^4 decomposes into a one-dimensional and 2 three-dimensional representations.*

Denote by $G = Aut(M^7, \omega^3)$ the connected component of the automorphism group of the nearly parallel G_2-manifold. The isotropy subgroup $G(m)$ of any point $m \in M^7$ is a subgroup of G_2. Thus, we obtain

$$\dim(G) - \dim(G(m)) \leq 7, \quad G(m) \subset G_2.$$

Theorem 6.8. *Let (M^7, ω^3) be a simply connected, compact manifold with nearly parallel G_2-structure not isometric to the sphere S^7. Then the automorphism group G has dimension ≤ 13.*

Proof. First we discuss the case of $15 \leq \dim(G)$. Then the isotropy subgroup $G(m)$ is a subgroup of G_2 with $8 \leq \dim(G(m))$. However, the group G_2 contains only two subgroups satisfying this condition, namely $G(m) = SU(3)$ and G_2 (see [12]). If $G(m) = G_2$ for any point $m \in M^7$, the Weyl tensor vanishes identically and the space M^7 is the sphere S^7. Suppose that there exists a point $m \in M^7$ such that $G(m) = SU(3)$. Then the group G acts transitively on M^7. Moreover, G is a simply connected, compact group of dimension 15 containing a subgroup isomorphic to $SU(3)$. The classification of compact groups yields that there exists only one group with these properties, namely $G = SU(4)$. The Riemannian metric on M^7 is given by an $SU(3)$-invariant scalar product of $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}^1$. The family of $SU(3)$-invariant scalar products depends on one positive parameter, but only the usual scalar product in $\mathbb{R}^7$ defines an Einstein metric on the homogeneous space $M^7 = SU(4)/SU(3)$. Consequently, M^7 is isometric to the sphere S^7.

Next we study the case of $\dim(G) = 14$. Then $7 \le \dim(G(m))$ for any point $m \in M^7$. The group G_2 does not contain a subgroup of dimension seven (see [12]) and therefore we obtain again $G(m) = SU(3)$ or G_2. The case $G(m) = G_2$ for any point $m \in M^7$ is impossible. Suppose $G(m) = SU(3)$ for some point. Then G is a compact group of dimension 14 containing a subgroup isomorphic to $SU(3)$. Moreover, G acts on M^7 with cohomogeneity one. Since M^7 is simply connected, there exists a point $m_0 \in M^7$ such that $G(m_0) = G$. Then G is isomorphic to G_2. In a neighbourhood of this point the Einstein metrics is a warped product metric $dr^2 \oplus f(r)g_0$, where g_0 is a G_2-invariant metric on the sphere $G_2/SU(3) = S^6$. Since the metric is regular at the point m_0, M^7 is a space of constant sectional curvature (see [4]). □

Theorem 6.9. *Let (M^7, ω^3) be a simply connected, compact manifold with nearly parallel G_2-structure not isometric to the sphere S^7. The group SU(3) cannot occur as an isotropy subgroup $G(m) \subset Aut(M^7, \omega^3)$.*

Proof. The isotropy group $G(m)$ of an arbitrary point $m \in M^7$ is a subgroup of G_2. Suppose that it is isomorphic to $SU(3)$ for one point $m \in M^7$. The isotropy representation $G(m) \longrightarrow SO(T_m(M^7))$ is the standard representation of $SU(3)$ in $SO(7)$. The possible dimensions of $G(m)$-invariant subspaces $V \subset T_m(M^7)$ are 0, 1, 6 and 7. The tangent space $T_m(N)$ of the orbit $N = G \cdot m = G/G(m)$ defines a $G(m)$-invariant subspace. Consequently, we obtain four possibilities:

(a) $G = G(m) = SU(3)$;

(b) $\dim(G) = 9$ and $G(m) = SU(3)$;

(c) $\dim(G) = 14$;

(d) $\dim(G) = 15$.

In case $\dim(G) = 14$ or 15, M^7 is isometric to the sphere S^7. If $\dim(G) = 9$, the automorphism group G is (locally) isomorphic to $G = SU(3) \times U(1)$. Denote by X the Killing vector field corresponding to the $U(1)$-action. Suppose that X has a zero point m^* and consider the orbit N through m^*. Then X vanishes at every point of N and therefore by Corollary 6.4, N is a one- or three-dimensional submanifold. The group $SU(3)$ acts on N as a group of isometrie s and we obtain an isomorphism $SU(3) \longrightarrow \mathrm{Iso}(N)$. The compact group $\mathrm{Iso}(N)$ is isomorphic to $U(1)$ (in case $\dim(N) = 1$) or to $SO(4)$ (in case $\dim(N) = 3$). Since any two- or four-dimensional real representation of the group $SU(3)$ is trivial, we conclude that G acts trivially on N. Hence, G is a nine-dimensional subgroup of G_2, a contradiction. Consequently, the Killing vector field X corresponding to the $U(1)$-action has constant length one. Next we prove that the $U(1)$-action on M^7 is a free action. Indeed, for any point m^* the isotropy subgroup $G(m^*)$ of $G = SU(3) \times U(1)$ has the dimension bounded by $\dim(G) - 7 = 2 \le \dim(G(m^*))$. In case $\dim(G(m^*)) = 2$, the group G acts transitively on M^7 and then the isotropy group $G(m) = SU(3)$ cannot occur. Hence, $G(m^*) = G_1 \times \mathbb{Z}_p \subset SU(3) \times U(1)$ is a group of dimension at least three. There are only two possibilities: $G_1 = SU(2)$ or $G_1 = SU(3)$. In both cases we get a $\mathbb{Z}_p$-action preserving the orientation on the six-dimensional sphere $S^6 = G_2/SU(3)$ commuting with the usual $SU(3)$-action on S^6. This means that the group $\mathbb{Z}_p$ is trivial, i.e. the action of $U(1)$ is free.

This $U(1)$-action on M^7 defines a compact six-dimensional manifold $K^6 = M^7/U(1)$ as well as a principal bundle $\pi : M^7 \longrightarrow K^6$. Since M^7 is an Einstein space of positive scalar curvature, K^6 is also an Einstein space of positive scalar curvature. The group $SU(3)$ acts as a group of isometries on K^6 and the isotropy subgroups of this action are $SU(2)$ or $SU(3)$. Hence K^6 is isometric to the projective space $\mathbb{C}P^3$. Moreover, the 2-form $\mathrm{d}X$ is a horizontal 2-form

$$\begin{aligned}(X \lrcorner \mathrm{d}X)(Y) = \mathrm{d}X(X,Y) &= X\langle X,Y\rangle - Y\langle X,X\rangle - \langle X,[X,Y]\rangle \\ &= \langle \nabla_X X, Y\rangle + \langle X, \nabla_Y X\rangle = 0,\end{aligned}$$

i.e. X is a connection in the principal $U(1)$-fibre bundle $\pi : M^7 \longrightarrow K^6$ with curvature form $\mathrm{d}X$. Finally it turns out that M^7 is the seven-dimensional sphere.

It remains to discuss the case of $\dim(G) = 8$. In this case, the group G coincides with $G(m) = SU(3)$ and acts on M^7 with cohomogeneity two. The subgroups of $SU(3)$ and their dimensions are:

$SO(3)$	3
$S(U(2)\times U(1))$	4
$SU(2)$	3
$U(1)\times U(1)$	2
$U(1)$	1

The orbit $G/G(m^*)$ for any point $m^* \in M^7$ is therefore either a point or at least a four-dimensional submanifold. The group $G(m^*) = S(U(2)\times U(1))$ cannot occur since the Euler characteristic of $G/G(m^*) = SU(3)/S(U(2)\times U(1)) = \mathbb{C}P^2$ is not zero (Theorem 6.6). On the other hand, near the point $m \in M^7$ all orbits are of type $SU(3)/SU(2)$. Since the set of all principal orbits of the G-action is dense, the type of the principal orbit is $G(m^*) = SU(2)$. Consequently, we see that $G = SU(3)$ acts on M^7 with two orbit types only. There is a finite set $\gamma_1, \ldots, \gamma_k$ of closed geodesics in M^7 such that $G(m_i) = SU(3)$ $(m_i \in \gamma_i)$ and any other orbit is of type $SU(3)/SU(2) = S^5$. There exists only one geodesic γ. Indeed, $Y = M^7/SU(3)$ is a two-dimensional manifold with k boundary components and

$$M^7 - \{\gamma_1, \ldots, \gamma_k\} \longrightarrow Int(Y)$$

is an S^5-fibration. On the other hand, we have

$$0 = \pi_1(M^7) = \pi_1(M^7 - \{\gamma_1, \ldots, \gamma_k\}) = \pi_1(Int(Y))$$

and $Y = D^2$ has only one boundary component. Consequently, M^7 is a seven-dimensional Einstein manifold with isometry group $SU(3)$ and the principal orbits are of type $SU(3)/SU(2)$; there exists only one exceptional orbit – the fixed-point set of $SU(3)$. It turns out that M^7 is isometric to the standard seven-dimensional sphere S^7 (M^7 is topologically the sphere and the metric is an $SU(3)$-invariant Einstein metric with respect to the usual action of $SU(3) \subset SO(6) \subset SO(8)$ see [3]). □

Corollary 6.10. *Let (M^7, ω^3) be a simply connected, compact manifold with nearly parallel G_2-structure not isometric to the sphere S^7. Then*
(i) *the dimension of the automorphism $G = Aut(M^7, \omega^3)$ has dimension ≤ 13;*
(ii) *any isotropy subgroup $G(m)$ has dimension ≤ 6.*

7. Nearly parallel G_2-structures with large symmetry group

In this section we will classify all seven-dimensional compact, simply connected manifolds with a nearly parallel G_2-structure and symmetry group of dimension at least 10. This classification includes in particular the classification of compact, simply connected homogeneous nearly parallel G_2-structures.

Let (M^7, g) be a compact, simply connected seven-dimensional nearly parallel G_2-manifold different from the sphere S^7. Let G be the connected component of the automorphism group of the G_2-structure. We already know that

$$\dim(G) \leq 13 \quad \text{and} \quad \dim(G(m)) \leq 6 \ \text{ for any point } m \in M^7$$

holds. We will discuss the spaces case by case depending on the dimension of the group G.

Case 1: $\dim(G) = 13$. In this case the dimension of the isotropy group $G(m)$ is six for any point $m \in M^7$ and the group G acts transitively on $M^7 = G/G(m)$. There exists only one connected six-dimensional subgroup of the group G_2 (see [12]), namely the isotropy group of the exceptional orbit of the G_2-action on the Grassmannian manifold $G_3(\mathbb{R}^7)$ (see Proposition 2.6). The Lie algebra of this subgroup is defined by the relations:

$$\begin{aligned} &\omega_{12} + \omega_{34} + \omega_{56} = 0, \quad \omega_{17} + \omega_{36} + \omega_{45} = 0, \quad \omega_{27} + \omega_{35} - \omega_{46} = 0, \\ &\omega_{13} = \omega_{14} = \omega_{15} = \omega_{16} = \omega_{23} = \omega_{24} = \omega_{25} = \omega_{26} \\ &\quad = \omega_{37} = \omega_{47} = \omega_{57} = \omega_{67} = 0 \end{aligned}$$

and the subgroup is isomorphic to $G(m) = SO(4) = [SU(2) \times SU(2)]/\{\pm 1\}$. Denote by G^* and $G^*(m)$ the 2-fold covering of the group G, respectively, of the group $G(m)$. Then G^* is a compact, simply connected 13-dimensional Lie group containing a subgroup isomorphic to $G^*(m) = Sp\,(1) \times Sp\,(1)$. Using the classification of simple Lie groups we deduce that G^* is isomorphic to $G^* = Sp\,(2) \times Sp(1)$. Consequently, the homogeneous Einstein manifold M^7 is of type $M^7 = [Sp\,(2) \times Sp\,(1)]/[Sp\,(1) \times Sp\,(1)]$ and therefore M^7 is isometric either to the standard sphere S^7 or to the squashed sphere S^7_{squas}.

Case 2: $\dim(G) = 12$. In this case the dimension of any isotropy group $G(m)$ is bounded by $5 \leq \dim(G(m)) \leq 6$. Since the group G_2 does not contain a subgroup of dimension five we obtain that any isotropy group $G(m)$ has dimension six, i.e. any isotropy group is a six-dimensional subgroup of G_2 containing the group $SO(4)$ described above:

$$SO(4) \subset G(m) \subset G_2.$$

It is a matter of fact that such a subgroup of G_2 coincides with $SO(4)$. Indeed, consider the covering $G_2/SO(4) \longrightarrow G_2/G(m)$. Any deck transformation $g \in G_2$ is homotopic to the identity map and therefore its Lefschetz number coincides with the Euler characteristic $\chi(G_2/SO(4)) > 0$, a contradiction. Consequently, the group G acts on M^7 with one orbit type only and M^7 is the total space of a fibration over S^1 with the fibre $F = G/SO(4)$. On the other hand, the exact homotopy sequence of this fibration yields

$$\cdots \longrightarrow \pi_1(F) \longrightarrow \pi_1(M^7) = 1 \longrightarrow \pi(S^1) = \mathbb{Z} \longrightarrow \pi_0(F) = 1,$$

a contradiction. Finally we see that the case $\dim(G) = 12$ is impossible.

Case 3: $\dim(G) = 11$. In this case the dimension of any isotropy group $G(m)$ is bounded by $4 \leq \dim(G(m)) \leq 6$.

Suppose that $\dim(G(m)) = 4$ for one point $m \in M^7$. Then G acts transitively, $M^7 = G/G(m)$, and the isotropy group $G(m) \subset G_2$ is connected. Using the list of all connected subgroups of the exceptional group G_2 (see [12]) we obtain two possibilities:

(a) $G(m)$ is the subgroup $[SU(2) \times U(1)]\{\pm 1\}$ of $SU(3)$. This is in fact the group $SU(3) \cap SO(4)$ and its Lie algebra is given by the equations:

$$\omega_{12} + \omega_{34} + \omega_{56} = 0, \qquad \omega_{36} + \omega_{45} = 0, \qquad \omega_{35} - \omega_{46} = 0,$$

$$\omega_{13} = \omega_{14} = \omega_{15} = \omega_{16} = \omega_{17} = \omega_{23} = \omega_{24} = \omega_{25} = \omega_{26} = \omega_{27} = 0,$$

$$\omega_{37} = \omega_{47} = \omega_{57} = \omega_{67} = 0.$$

The representation of $G(m)$ in $\mathbb{R}^7$ splits into a one-, two- and four-dimensional invariant subspace,

$$\mathbb{R}^7 = E^1 \oplus E^2 \oplus E^4,$$

where $E^2 = Span(e_1, e_2)$, $E^4 = Span(e_3, e_4, e_5, e_6)$ and $E^1 = Span(e_7)$.

(b) $G(m)$ is the subgroup $[U(1) \times SU(2)]/\{\pm 1\}$ of $SO(4) = [SU(2) \times SU(2)]/\{\pm 1\}$. The Lie algebra of this group is given by the equations:

$$\begin{aligned}\omega_{13} &= \omega_{14} = \omega_{15} = \omega_{16} = \omega_{23} = \omega_{24} = \omega_{25} = \omega_{26}\\ &= \omega_{37} = \omega_{47} = \omega_{57} = \omega_{67} = 0,\end{aligned}$$

$$\omega_{12} + \omega_{34} + \omega_{56} = 0,$$

$$\omega_{17} + \omega_{36} + \omega_{45} = 0, \qquad \omega_{36} = \omega_{45}, \qquad \omega_{27} + \omega_{35} - \omega_{46} = 0, \qquad \omega_{35} = -\omega_{46}.$$

The representation of $G(m)$ in $\mathbb{R}^7$ splits into a three- and four-dimensional invariant subspace,

$$\mathbb{R}^7 = F^3 \oplus F^4,$$

where $F^3 = Span(e_1, e_2, e_7)$ and $F^4 = Span(e_3, e_4, e_5, e_6)$.

First we consider the case that $\pi_1(G)$ is a finite group. Denote by G^* its universal covering and lift the isotropy subgroup to $G^*(m) = Sp(1) \times U(1)$. Then G^* is a simply connected Lie group of dimension 11 containing the two-dimensional torus $T^2 \subset Sp(1) \times U(1)$.

Moreover, since the Euler characteristic of $M^7 = G^*/G^*(m)$ vanishes we conclude that the rank of G^* is greater or equal to 3,

$$rank(G^*) \geq 3\,, \quad \dim(G^*) = 11\,, \quad \pi_1(G^*) = 1.$$

The classification of all compact Lie groups yields that G^* is isomorphic to $SU(3) \times SU(2)$. In case (a) the isotropy group $G(m)$ is contained in $SU(3)$ and consequently the space M^7 admits two real Killing spinors (the G_2-structure is of type 2). On the other hand, the automorphism group of the G_2-structure of the manifold $M(3, 2)$ with two Killing spinors described in Section 4 is isomorphic to $SU(3) \times SU(2)$, this group acts transitively on M^7 and the isotropy representation coincides with the representation of $G(m)$ in case (a). Hence, in case (a) M^7 is isometric to $M(3, 2)$. In a similar way we can handle case (b). The manifold $N(1, 1)$ admits a G_2-structure of type 1 (not the 3-Sasakian metric!, see Sections 4 and 5) and the automorphism group of this G_2-structure coincides obviously with the isometry group $SU(3) \times SU(2)$. A calculation of the isotropy representation yields that it coincides with the representation of case (b) and consequently M^7 is isometric to $N(1, 1)$.

Suppose now that $\pi_1(G)$ is not a finite group. The exact homotopy sequence

$$\cdots \longrightarrow \pi_2(M^7) \longrightarrow \pi_1(G(m)) = \mathbb{Z} \longrightarrow \pi_1(G) \longrightarrow 1$$

yields that $\pi_1(G(m)) = \pi_1(G) = \mathbb{Z}$. Consider a finite covering G^* of G such that G^* splits into $G^* = U(1) \times G_1$, where G_1 is a simply connected group of dimension 10. Then G_1 is isomorphic to $Spin(5)$. The decomposition $G^* = U(1) \times Spin(5)$ defines a Killing vector field X on M^7 invariant with respect to the action of $Spin(5)$. Consequently, X has a constant length. In particular, at the point $m \in M^7$ the isotropy group $G(m)$ preserves the vector $X(m)$, i.e. the group $G(m)$ is of type $G(m) = SU(3) \cap SO(4)$ and the isotropy representation splits into

$$T_m(M^7) = E^1 \oplus E^2 \oplus E^4.$$

On the other hand, the embedding $\Phi : G^*(m) = U(1) \times Spin(3) \longrightarrow U(1) \times Spin(5) = G^*$ is given by two injective homomorphisms

$$i : Spin(3) \longrightarrow Spin(5)\,, \qquad j : U(1) \longrightarrow U(1)$$

$(\pi_1(G^*/G^*(m)) = 1!)$ and by one homomorphism $k : U(1) \longrightarrow Spin(5)$,

$$\Phi(z, g) = (j(z), k(z) \cdot i(g)).$$

Therefore the isotropy representation of the space $G^*/\Phi(G^*(m))$ considered only as a $Spin(3)$-representation is isomorphic to the isotropy representation of the space $Spin(5)/i(Spin(3))$. There are only two injective homomorphisms $i_1, i_2 : Spin(3) \longrightarrow Spin(5)$. The first of them i_1 is related to the five-dimensional irreducible representation of $SO(3)$ and i_2 is the usual inclusion of $SO(3)$ into $SO(5)$. In case of i_1 we obtain that the isotropy representation of the homogeneous space is irreducible and in case of i_2 we obtain the isotropy representation of the Stiefel manifold $V_{5,2}$ which splits into the irreducible subspaces $E^1 \oplus E^3 \oplus E^3$. This contradicts the mentioned decomposition of $T_m(M^7)$ and finally the case $G^* = U(1) \times Spin(5)$ is not possible.

We discuss now the case that any isotropy group $G(m)$ is a six-dimensional group, i.e. $G(m) = SO(4) \subset G_2$. Since $\dim(G) = 11$, any orbit $N = G/G(m) \subset M^7$ has dimension five and its tangent space $T_m(N) \subset T_m(M^7)$ defines a $G(m) = SO(4)$-invariant subspace of $T_m(M^7) = \mathbb{R}^7$. The representation of the group $SO(4) \subset G_2 \subset SO(7)$ splits into two $SO(4)$-irreducible parts, namely $\mathbb{R}^7 = F^3 \oplus F^4$ where $F^3 = Span(e_1, e_2, e_7)$ and $F^4 = Span(e_3, e_4, e_5, e_6)$, a contradiction. Consequently, this case is impossible.

Case 4: $\dim(G) = 10$. In this case the dimension of any group $G(m)$ is bounded by $3 \le \dim(G(m)) \le 6$.

Suppose that $\dim(G(m)) = 3$ for one point $m \in M^7$. Then G acts transitively, $M^7 = G/G(m)$, and the isotropy group $G(m) \subset G_2$ is connected. Using the list of all connected subgroups of the exceptional group G_2 (see [12]) we obtain four possibilities. In any case, $G(m)$ is isomorphic to $SO(3)$ or to $SU(2)$. Since $\pi_1(M^7) = 1$ we obtain $\pi(G) = \pi_1(G(m)) = 0$ or $\mathbb{Z}_2$. Consider the universal coverings G^* and $G^*(m) = Spin(3)$. Then G^* is a simply connected Lie group of dimension 10. Moreover, since the Euler characteristic of $M^7 = G/G^*(m)$ vanishes we conclude that the rank of G^* is greater or equal to 2,

$$rank(G^*) \ge 2\,, \quad \dim(G^*) = 10\,, \quad \pi_1(G^*) = 1.$$

The classification of all compact Lie groups yields that G^* is isomorphic to $Spin(5)$ and the manifold M^7 is isometric to the Stiefel manifold $V_{5,2}$ or to the spaces $SO(5)/SO(3)$ described in Section 4.

Suppose now that the isotropy group $G(m)$ is a four-dimensional subgroup for one point $m \in M^7$. Then $G(m)$ is one of the two subgroups of G_2 considered in the discussion of the case $\dim(G) = 11$. In particular, $G(m)$ is a connected subgroup. The orbit $G \cdot m$ through m is a six-dimensional manifold, but only the group $G(m) = SU(3) \cap SO(4) \subset G_2$ has a six-dimensional invariant subspace. Consequently, $G/G(m)$ is the principal orbit of the G-action on M^7 and there are no other orbits of dimension six. But exceptional orbits do not exist at all. Indeed, since $SU(3)$ cannot occur as an isotropy subgroup, an exceptional orbit must be of type $O^4 = G/SO(4)$. However, the isotropy representation of $SO(4)$ is $F^3 \oplus F^4$, a contradiction to Corollary 6.7. Finally the G-action defines a fibration $M^7 \longrightarrow M^7/G = S^1$ and the exact homotopy sequence yields that M^7 cannot be simply connected.

It remains to discuss the situation where any orbit is a four-dimensional manifold and every isotropy group $G(m)$ coincides with $SO(4)$. In this situation we can apply the same argument as before and we obtain again a contradiction to Corollary 6.7.

In particular we proved the following:

Theorem 7.1. *Any compact nearly parallel G_2-manifold with automorphism group of dimension* $\dim(G) \ge 10$ *is homogeneous.*

Probably there exist non-homogeneous nearly parallel G_2-manifolds admitting an automorphism group of dimension 9, 8, ... However, explicit non-homogeneous examples with a nine- or eight-dimensional automorphism group up to now are not known.

On the other hand, using similar arguments as before one can finish the classification of compact, homogeneous nearly parallel G_2-manifolds. It turns out that in case $\dim(G) \leq 9$ the space is isometric to $Q(1, 1, 1)$ or to one of the manifolds, $N(k, l)$.

Theorem 7.2. *Any compact, simply connected, homogeneous nearly parallel G_2-manifold is one of the spaces described in Tables* 1–3.

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